Mathematical Expectation

Data Science and A.I. Lecture Series

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- Expectation provides a measure of central tendency in probability distributions.
- Expectation is useful in both discrete and continuous probability distributions.
- Problems and examples help in understanding practical applications.

Objectives

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- Explore properties of expectation.
- Compute moments and other statistical measures.
- Apply addition and multiplication theorems of expectation.
- Solve problems on mathematical expectation.

Expectation of a Discrete Random Variable

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- Expectation represents the long-run average value of the random variable.

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$$E(X) = \int_{-\infty}^{\infty} xf(x)dx$$
- Expectation is also known as the mean of the probability distribution.
- The integral ensures that all possible values of X contribute to the expected value.

Example 1: Expected Value in a Game (Discrete)

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A game involves drawing a card from a box containing cards numbered 1, 3, and 5. If a card numbered x is drawn, the player wins $2x$ rupees. Find the expected winnings.

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Solution

The probability of drawing each card is equal: $P(X = 1) = P(X = 3) = P(X = 5) = \frac{1}{3}$.

$$E(X) = 2(1) \times \frac{1}{3} + 2(3) \times \frac{1}{3} + 2(5) \times \frac{1}{3}$$

$$E(X) = \frac{2}{3} + \frac{6}{3} + \frac{10}{3} = \frac{18}{3} = 6$$

Example 2: Expected Revenue (Discrete)

Example

A fruit vendor sells mangoes for Rs 20 each. If the daily demand follows the probability distribution $P(X = 10) = 0.3$, $P(X = 15) = 0.5$, $P(X = 20) = 0.2$, find the expected daily revenue.

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Solution

$$E(X) = 20(10) \times 0.3 + 20(15) \times 0.5 + 20(20) \times 0.2$$
$$E(X) = 60 + 150 + 80 = 290$$

Example 3: Expected Time for a Task (Continuous)

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Solution

$$E(X) = \int_0^1 xf(x)dx = \int_0^1 x(2x)dx$$

$$E(X) = 2\int_0^1 x^2 dx = 2 \times \frac{1}{3} = \frac{2}{3}$$

Example 4: Expected Production Output (Continuous)

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A machine produces items with output per hour X following the density function $f(x) = 3(1-x)^2$ for $0 < x < 1$. Find the expected output.

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Solution

$$E(X) = \int_0^1 xf(x)dx = \int_0^1 x(3(1-x)^2)dx$$

Expanding and solving,

$$E(X) = 3\int_0^1 x(1-2x+x^2)dx = 3\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1$$

$$E(X) = 3\left[\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\right] = 3\left[\frac{6}{12} - \frac{8}{12} + \frac{3}{12}\right] = 3 \times \frac{1}{12} = \frac{1}{4}$$

Properties of Expectation of One-Dimensional Random Variable

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Properties of mathematical expectation of a random variable X:

- $E(k) = k$, where k is a constant.
- $E(kX) = kE(X)$, where k is a constant.
- $E(aX + b) = aE(X) + b$, where a and b are constants.

Expectation of a Random Variable

Let X be a discrete random variable taking values $x_1, x_2, x_3, \ldots$ with probabilities $p_1, p_2, p_3, \ldots$.

- By definition of expectation:

$$E[k] = \sum_i k p_i \tag{1}$$

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$$E[aX + b] = \sum_i (a x_i + b) p_i \tag{4}$$

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- Expanding the sum:

$$E[aX + b] = a \sum_i x_i p_i + b \sum_i p_i = aE[X] + b \tag{5}$$

Expectation in the Continuous Case

Let X be a continuous random variable with probability density function $f(x)$.

- Expectation of a constant:

$$E[k] = \int_{-\infty}^{\infty} kf(x)dx = k \cdot 1 = k \tag{6}$$

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- Splitting the integral:

$$E[aX + b] = a\int_{-\infty}^{\infty} xf(x)dx + b\int_{-\infty}^{\infty} f(x)dx \tag{9}$$

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- Final result:

$$E[aX + b] = aE[X] + b \tag{10}$$

Example: Expectation Calculation

Given the following probability distribution:

X	$p(X)$
-2	0.15
-1	0.30
0	0
1	0.30
2	0.25

Find:

- $E(X)$

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- $E(2X + 3)$

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Find:

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- $E(2X + 3)$
- $E(X^2)$

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Find:

- $E(X)$
- $E(2X+3)$
- $E(X^2)$
- $E(4X-5)$

Solution: Expectation of X

- By definition:

$$E(X) = \sum_i x_i p_i \tag{11}$$

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- Substituting values:

$$(-2)(0.15) + (-1)(0.30) + (0)(0) + (1)(0.30) + (2)(0.25) \tag{12}$$

Solution: Expectation of X

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$$E(X) = \sum_i x_i p_i \tag{11}$$

- Substituting values:

$$(-2)(0.15) + (-1)(0.30) + (0)(0) + (1)(0.30) + (2)(0.25) \tag{12}$$

- Simplifying:

$$-0.3 - 0.3 + 0 + 0.3 + 0.5 = 0.2 \tag{13}$$

Solution: Expectation of $2X + 3$

- Using linearity:

$$E(2X + 3) = 2E(X) + 3 \tag{14}$$

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- Using linearity:
$$E(2X + 3) = 2E(X) + 3 \tag{14}$$
- Substituting $E(X) = 0.2$:
$$2(0.2) + 3 = 0.4 + 3 = 3.4 \tag{15}$$

Solution: Expectation of X^2

- By definition:

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- Substituting values:

$$(4)(0.15) + (1)(0.30) + (0)(0) + (1)(0.30) + (4)(0.25) \tag{17}$$

Solution: Expectation of X^2

- By definition:

$$E(X^2) = \sum_i x_i^2 p_i \tag{16}$$

- Substituting values:

$$(4)(0.15) + (1)(0.30) + (0)(0) + (1)(0.30) + (4)(0.25) \tag{17}$$

- Simplifying:

$$0.6 + 0.3 + 0 + 0.3 + 1 = 2.2 \tag{18}$$

Solution: Expectation of $4X - 5$

- Using linearity:

$$E(4X - 5) = 4E(X) - 5 \tag{19}$$

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- Using linearity:

$$E(4X - 5) = 4E(X) - 5 \tag{19}$$

- Substituting $E(X) = 0.2$:

$$4(0.2) - 5 = 0.8 - 5 = -4.2 \tag{20}$$

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- Stated and proved addition and multiplication theorems.
- Solved problems for both discrete and continuous cases.

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