Solutions for Questions Based on Central Limit Theorem (CLT) and Uniformly Minimum Variance Unbiased Estimator (UMVUE)

Selected Topics and Numericals on Probability and Statistics

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Question 1

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- (d) Degenerate at 0

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- CLT states:

$$\frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \xrightarrow{d} N(0,1)$$

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- Thus, the correct answer is **(c)** $N(0, 2\sigma^2)$.

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- Expectation:

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- S^2 is a biased estimator for σ^2.
- To correct this bias, we use:

$$\hat{\sigma}^2 = \frac{n}{n-1}S^2 = \frac{1}{n-1}\sum_{i=1}^{n} X_i^2$$

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